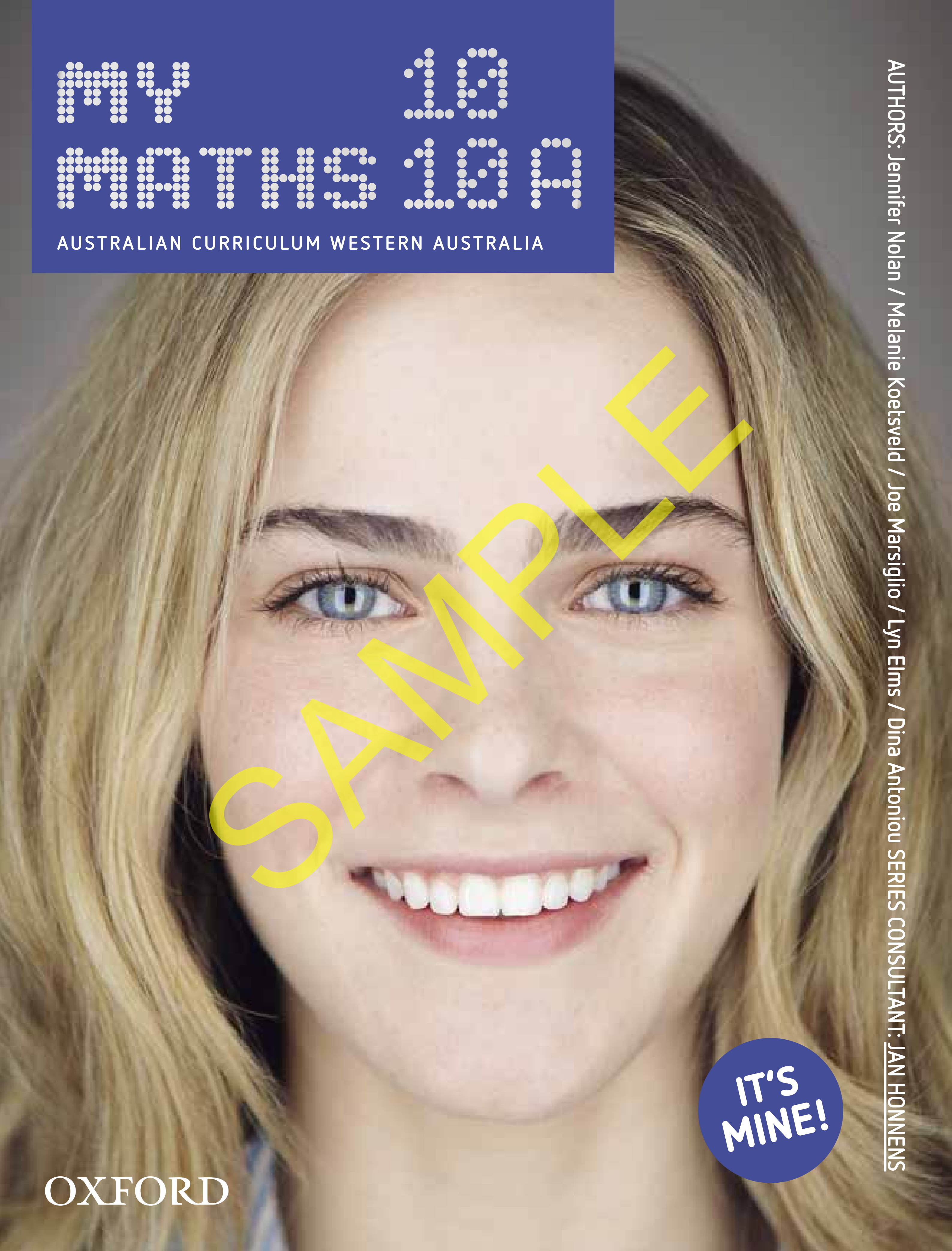
MY MATHS 10 10A

AUSTRALIAN CURRICULUM WESTERN AUSTRALIA

AUTHORS: Jennifer Nolan / Melanie Koetsveld / Joe Marsiglio / Lyn Elms / Dina Antoniou SERIES CONSULTANT: JAN HONNENS



IT'S MINE!

OXFORD

NUMBER AND ALGEBRA

1 2 3 4 5 6 7 8 9 10 10A

MEASUREMENT AND GEOMETRY

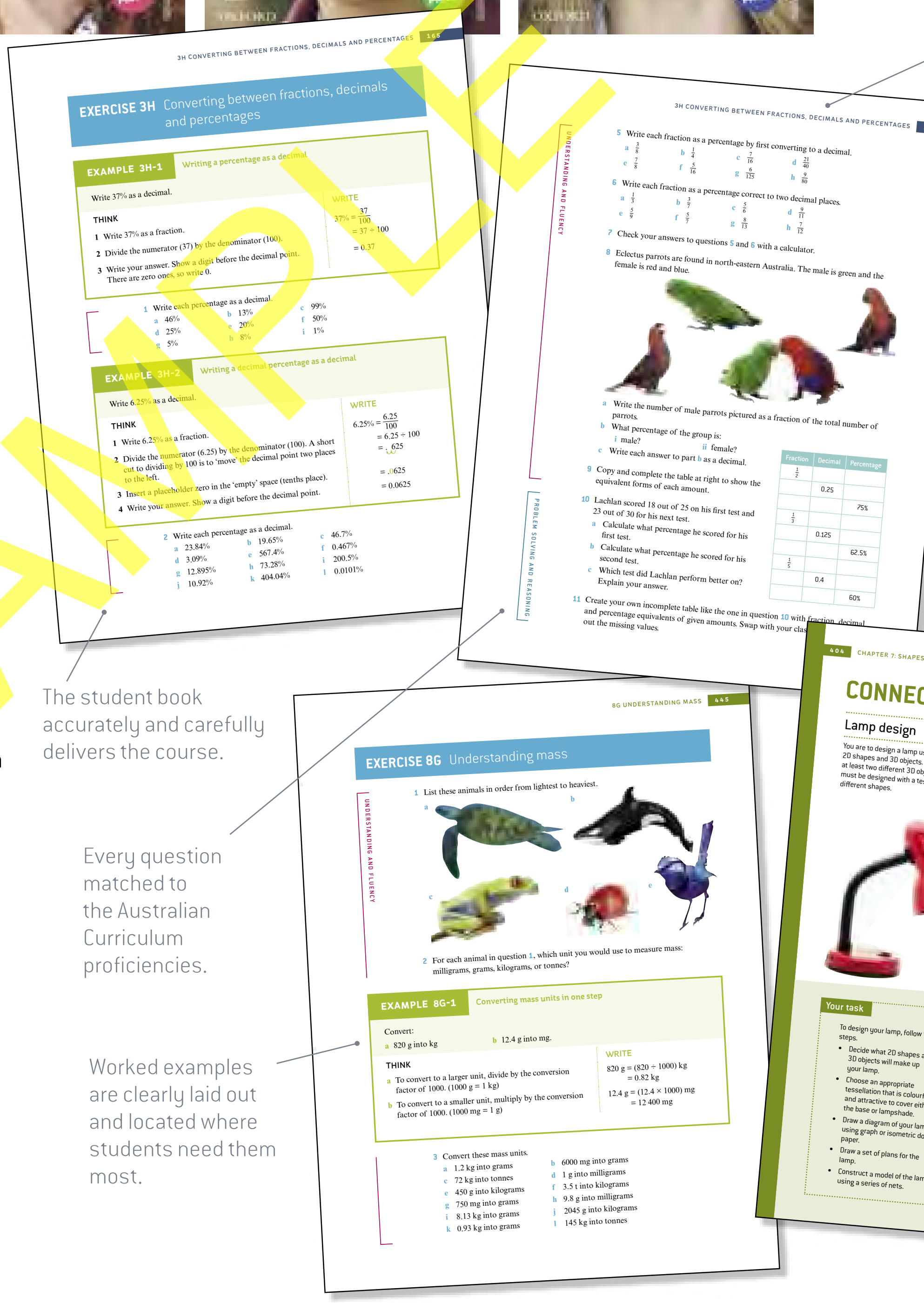
STATISTICS AND PROBABILITY

OXFORD MYMATHS FOR WESTERN AUSTRALIA

Oxford MyMaths for Western Australia has been specifically developed to support students wherever and whenever learning happens: in class, at home, with teacher direction or in independent study.

STUDENT BOOK + OBOOK/ASSESS

- Finely levelled exercises to ensure smooth progress
- Integrated worked examples – right where your students need them
- Learning organised around the 'big ideas' of mathematics
- Discovery, practice, thinking and problem-solving activities promote deep understanding
- A wealth of revision material to consolidate and prove learning
- Rich tasks to apply understanding
- Highly accessible and easy to navigate
- Comprehensive digital tutorials and guided examples to support independent progress



The student book accurately and carefully delivers the course.

Every question matched to the Australian Curriculum proficiencies.

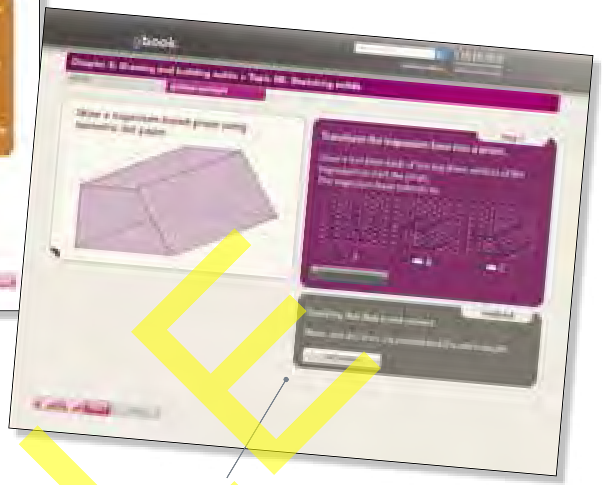
Worked examples are clearly laid out and located where students need them most.

24/7 LEARNING AND SUPPORT

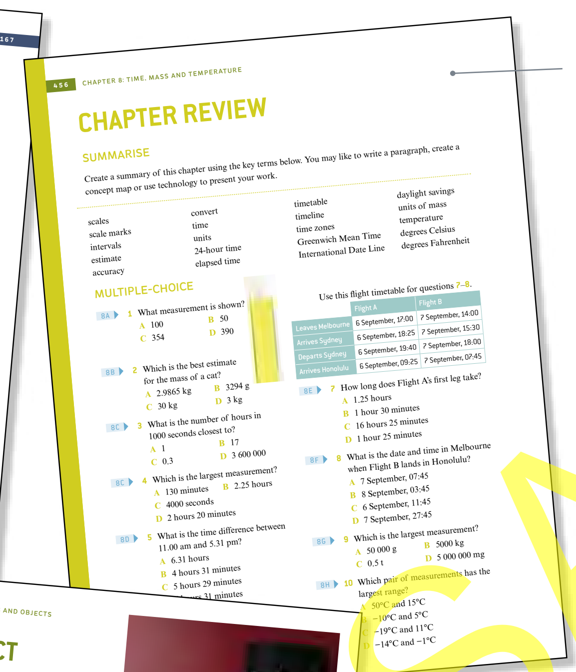
E-tutors scaffold understanding of key concepts and build confidence.

Self-discovery opportunities for students through guided exploration.

Guided examples support practice and fluency

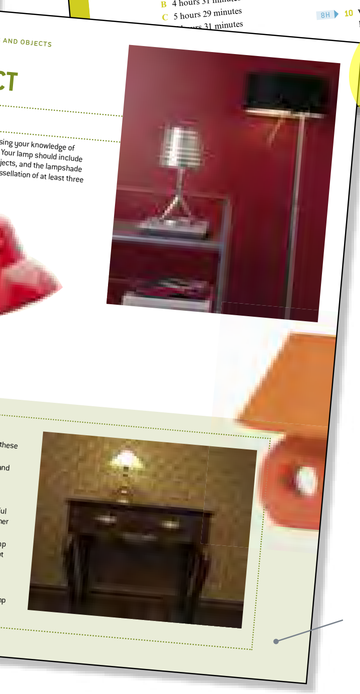


Finely levelled content enables students to progress with ease



Ample revision to consolidate understanding and prove that learning has happened

Students receive feedback for incorrect responses



Optimise understanding and performance.

Personalised learning: tailor the very best learning experiences for all.

Rich tasks where students can demonstrate understanding

Intervention and extension worksheets supplied for every topic.

TEACHER OBOOK/ASSESS

Practical classroom resources and tools:

- Manage student differentiation
- Correct common misconceptions
- Assign work
- Set tests
- Monitor results
- Any device, anytime, anywhere.

NUMBER AND ALGEBRA

6 POLYNOMIALS

ESSENTIAL QUESTION

What are polynomials and how are they useful when modelling a relationship?

Are you ready?

6A 10A

1 Simplify each expression.

a $4x^2 - 7x + 3 + 2x^2 - 5 - x$

b $(3x^2 - 5x) - (x^2 + 6x)$

c $(2x^3 + x^2 + 1) - (5x^3 - 4x^2 - 3x)$

d $(x^3 - 2x^2 + 3x - 4) - (x^3 + 5x^2 - 2x - 9)$

6A 10A

2 Simplify each expression.

a $3x \times x^2$ b $7x^3 \times 2x^2$

c $5x^2 \div x$ d $24x^3 \div 4x$

6A 10A

3 Expand and simplify each product.

a $x^2(3x^3 - 5)$ b $(x - 4)(x + 7)$

c $2(x + 3)^2$ d $x(3x - 2)(2x + 1)$

6A 10A

4 Evaluate $2x^3 - 3x^2 - 4x + 3$ after substituting each x value.

a $x = 1$ b $x = 3$

c $x = 0$ d $x = -2$

6B 10A

5 Consider this long division problem.

$$\begin{array}{r} 253 \\ 13\overline{)3296} \\ \underline{26} \\ 69 \\ \underline{65} \\ 46 \\ \underline{39} \\ 7 \end{array}$$

a The dividend is:

A 7 B 13

C 253 D 3296

b The divisor is:

A 7 B 13

C 253 D 3296

c The quotient is:

A 7 B 13

C 253 D 3296

d The remainder is:

A 7 B 13

C 253 D 3296

6B 10A

6 Use long division to find the quotient and remainder for:

a $456 \div 18$ b $2783 \div 23$

6D 10A

7 Solve each equation.

a $2x(x - 7) = 0$

b $x^2 - 3x - 18 = 0$

c $x^2 - 16 = 0$

d $x^2 + 4x - 1 = 0$

e $3x^2 - 6x + 3 = 0$

f $25 + x^2 = 0$

6E 10A

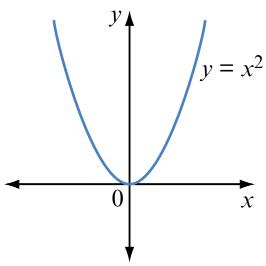
8 Find the x- and y-intercepts for each graph with the given rule.

a $y = 2x - 6$ b $y = x^2 + x - 12$

c $y = x^2 - 9$ d $y = x^2 + 4x + 4$

6F 10A

9 Describe the transformation/s to be performed on the graph of $y = x^2$ to produce the graph of:



a $y = x^2 + 4$

b $y = (x - 2)^2$

c $y = 3x^2$

d $y = -x^2$

e $y = (x + 1)^2 - 2$

f $y = -2(x - 5)^2 + 3$

6F 10A

10 Write the coordinates of the turning point for each new graph in question 9.

6A Understanding polynomials

Start thinking!

A **polynomial** is an expression with terms containing one variable only, and with that variable raised to a power that is a positive integer or zero.

1 Consider the quadratic expression $3x^2 - 5x + 7$.

- **a** What variable is used in the expression?
- **b** What is the power of x in:
  - **i** the first term?
  - **ii** the second term?
  - **iii** the third term? (Hint: what does x^0 equal?)
- **c** Is the expression a polynomial? Explain.
- **d** What is the highest power of x used? This is the **degree** of the polynomial.
- **e** Which term contains the highest power of x? This is the **leading term** of the polynomial.
- **f** What is the coefficient of the leading term? This is the leading **coefficient** of the polynomial.
- **g** Which term is the constant term?

2 Decide if each expression is a polynomial. Give a reason for your answer.

- **a** $x^3 + 7x^2 + 3x + 2$
- **b** $8x^2 - 2x^{\frac{1}{4}}$
- **c** $4x + 9x^4 - x^2$
- **d** $\frac{6}{x} + 5x^7$
- **e** $2x^5 - x^4 + 1 - \sqrt{x}$
- **f** $3x^2 + 2xy - y^3$

KEY IDEAS

- A polynomial is an expression that contains only one variable, such as x. Each term contains the variable raised to a non-negative integer value (0, 1, 2, 3, …).
- A polynomial can be written as $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x^1 + a_0x^0$ where n is the highest power and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are coefficients. $P(x)$ is read as 'P of x' meaning the polynomial P using the variable x.
- The degree of a polynomial (n) is the highest power of the variable. Polynomials are given names according to their degree. Some common polynomials are listed in the table at right.
- The leading term (a_nx^n) contains the highest power of the variable and is usually written first.
- To add or subtract polynomials, add or subtract any like terms.
- To expand the product of two polynomials, multiply each term in the first polynomial by each term in the second. If there are more than two polynomial factors to multiply together, expand two factors and then multiply the result by the remaining factors, one at a time.

Degree	Name	Example
0	constant	5 (or $5x^0$)
1	linear	$2x + 9$ (or $2x^1 + 9$)
2	quadratic	$-x^2 + 3x - 1$
3	cubic	$4x^3 - x^2 + 2x + 7$
4	quartic	$\frac{1}{2}x^4 + x^2 - 2$

EXERCISE 6A Understanding polynomials

1 Decide if each expression is a polynomial. For each polynomial, give its name as constant, linear, quadratic, cubic or quartic.

a $4x^3 + 2x^2 + 5x + 1$ b $1 - x^3y^2 + 7x$ c $6 - 9x$

d $3x + x^2 - 4$ e $2x^3 + 3\sqrt{x} - 6$ f 8

g $5x^3 - 2x^{\frac{1}{5}} + 7$ h $x^4 + x + 1$ i $\frac{3}{7}x^3$

EXAMPLE 6A-1 Identifying features of a polynomial

For the polynomial $2x^3 - x^2 + 7x + 4$, identify:

a the number of terms b the degree of the polynomial

c the constant term d the leading term

e the leading coefficient f the coefficient of the x^2 term

THINK	WRITE
a Terms are separated by + and − signs.	a There are four terms.
b Look for the highest power of x.	b Degree is 3.
c Look for a term without a variable (power of variable is 0).	c Constant term is 4.
d Look for the term that has the highest power of x.	d Leading term is $2x^3$.
e Write the coefficient of the leading term.	e Leading coefficient is 2.
f Write the coefficient of the term containing x^2. The sign shown between it and the term before belongs to the x^2 term. ($-x^2$ means $-1x^2$.)	f Coefficient of x^2 is -1.

UNDERSTANDING AND FLUENCY

2 For each polynomial below, identify:

i the number of terms ii the degree of the polynomial

iii the constant term iv the leading term

v the leading coefficient vi the coefficient of the x^2 term

a $2x^3 + 3x^2 + 4x + 5$ b $4x^5 + x^4 + 7x^3 - 2x^2 + 9x - 3$

c $-5x^4 - 2x^3 + 5x^2 + 1$ d $x^7 + x^6 - x^5 + x^4 + x^3 - x^2 + x$

e $9 - 3x - 6x^3 + 2x^2 - 7x^6$ f $3 - 11x^{10} + 5x^8$

3 Simplify each polynomial by collecting like terms.

a $4x^3 + 2x^2 - 4x - 1 + x^3 - 5x^2 + 3$ b $x^5 - 3x^2 + x^3 - 2x^4 - x^3 + 7x^2$

c $x^4 + 2 + 4x^2 + x^4 - 2x^2 - 5 + 6x^3$ d $(2x^2 - 5x + 1) + (3x^2 - 6x + 8)$

e $(5x^3 - 2x + 1) - (2x^3 - 7x + 4)$ f $(3x^2 + x + x^3) - (4x^4 - 3x^2 + 5x^3)$

EXAMPLE 6A-2 Evaluating a polynomial

If $P(x) = 2x^3 - x^2 + 7x + 4$, evaluate: **a** $P(3)$ **b** $P(0)$ **c** $P(-1)$

THINK	WRITE
a Substitute $x = 3$ into the expression and evaluate.	**a** $P(3) = 2(3)^3 - (3)^2 + 7(3) + 4$ $= 54 - 9 + 21 + 4$ $= 70$
b Substitute $x = 0$ into the expression and evaluate.	**b** $P(0) = 2(0)^3 - (0)^2 + 7(0) + 4$ $= 0 - 0 + 0 + 4$ $= 4$
c Substitute $x = -1$ into the expression and evaluate.	**c** $P(-1) = 2(-1)^3 - (-1)^2 + 7(-1) + 4$ $= -2 - 1 - 7 + 4$ $= -6$

4 If $P(x) = x^3 - 2x^2 + 5x - 8$, evaluate:

a $P(2)$ **b** $P(0)$ **c** $P(3)$ **d** $P(-3)$

5 If $P(x) = 3x^4 - 4x^3 + x^2 - 6x + 1$, evaluate:

a $P(3)$ **b** $P(1)$ **c** $P(-1)$ **d** $P(-2)$

6 If $P(x) = 2x^3 - x^2 + 3x - 6$ and $Q(x) = 3x^2 - 7x + 2$, find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $Q(x) - P(x)$ **d** $3P(x)$

e $-2Q(x)$ **f** $2P(x) + Q(x)$ **g** $P(x) - 4Q(x)$ **h** $3P(x) - 2Q(x)$

i $2P(-1)$ **j** $-3Q(2)$ **k** $P(3) + Q(3)$ **l** $5P(0) - 4Q(0)$

EXAMPLE 6A-3 Expanding the product of two polynomials

Expand and simplify each product.

a $2x^3(x^2 - 3x + 4)$ **b** $(x^2 - 2)(x^3 - x + 5)$ **c** $(2x^4 - x^2 + 3)(x^3 + 4x - 1)$

THINK	WRITE
a Multiply the term outside the brackets with each term inside the pair of brackets.	**a** $2x^3(x^2 - 3x + 4)$ $= 2x^5 - 6x^4 + 8x^3$
b Multiply each term in the first pair of brackets with each term in the second pair of brackets. Simplify like terms.	**b** $(x^2 - 2)(x^3 - x + 5)$ $= x^2(x^3 - x + 5) - 2(x^3 - x + 5)$ $= x^5 - x^3 + 5x^2 - 2x^3 + 2x - 10$ $= x^5 - 3x^3 + 5x^2 + 2x - 10$
c Multiply each term in the first pair of brackets with each term in the second pair of brackets. Simplify like terms.	**c** $(2x^4 - x^2 + 3)(x^3 + 4x - 1)$ $= 2x^4(x^3 + 4x - 1) - x^2(x^3 + 4x - 1) + 3(x^3 + 4x - 1)$ $= 2x^7 + 8x^5 - 2x^4 - x^5 - 4x^3 + x^2 + 3x^3 + 12x - 3$ $= 2x^7 + 7x^5 - 2x^4 - x^3 + x^2 + 12x - 3$

7 Expand and simplify each product.

a $x^3(x^2 + 2x + 7)$
b $4x^2(x^2 - 5x + 2)$
c $-6x^3(x^4 - 4x^2 + 1)$
d $(x + 3)(x^2 - 2x + 4)$
e $(x^2 - 1)(x^2 + 3x - 7)$
f $(x^2 - 5)(x^3 - x + 3)$
g $(x^4 - 3x + 2)(x^3 + 4x^2 - 1)$
h $(2x^5 - 3x^2 + 1)(x^3 - 5x + 2)$

EXAMPLE 6A-4 Expanding the product of three polynomials

Expand and simplify $(x + 2)(x - 3)(x + 4)$.

THINK	WRITE
1 Multiply two of the factors and simplify. It is easier to choose the last two.	$(x + 2)(x - 3)(x + 4)$ $= (x + 2)(x^2 + 4x - 3x - 12)$ $= (x + 2)(x^2 + x - 12)$
2 Multiply the linear factor by the quadratic factor.	$= x(x^2 + x - 12) + 2(x^2 + x - 12)$ $= x^3 + x^2 - 12x + 2x^2 + 2x - 24$
3 Simplify like terms.	$= x^3 + 3x^2 - 10x - 24$

8 Expand and simplify each product.

a $3x(x + 2)(x + 3)$
b $-7x^2(x + 5)(x - 2)$
c $(x + 3)(x + 1)(x + 6)$
d $(x - 3)(x - 4)(x + 1)$
e $(3x - 2)(x - 5)(x - 2)$
f $(4x + 1)(2x + 7)(x - 4)$

9 Expand and simplify each expression.

a $(x^3 + 4)^2$
b $(x^2 + 3x)^2$
c $2x(x^2 - 2)^2$
d $(x^3 - 5x^2)^2$
e $(x + 2)^3$
f $(2x - 3)^3$
g $(x + 3)^4$
h $(1 - x)^4$

10 What is the maximum number of terms in a polynomial of degree 5?

11 What is the minimum number of terms in a polynomial of degree 3?

12 If $P(x)$ is a polynomial of degree n, what is:

a the maximum number of terms in $P(x)$?
b the minimum number of terms in $P(x)$?
c the degree of $2P(x)$?
d the degree of $[P(x)]^2$?

13 Find the value of k in $P(x) = x^3 - 2x^2 + 3x + k$, if $P(2) = 2$.

14 Find the value of k in $P(x) = 3x^4 + 2x^3 - kx - 5$, if $P(-1) = 7$.

15 Find the value of a and b in $P(x) = x^3 + ax^2 + bx + 1$, if $P(3) = 31$ and $P(-2) = -19$.

CHALLENGE

16 If $P(x) = x^3 - 3x^2 + 2x + 1$, write simplified expressions for:

a $P(a)$
b $P(-3a)$
c $P(a^2)$
d $P(-a^2)$
e $P(a + 2)$
f $P(2a - 1)$

17 Show that $(2x - 1)^4 = 16x^4 - 32x^3 + 24x^2 - 8x + 1$.

18 Simplify each expression.

a $(x^2 - 2)^3$
b $(x^3 + x - 1)^2$
c $(2x^3 + 1)^4$

Reflect

What makes an algebraic expression a polynomial?

6B Division of polynomials

Start thinking!

So far you have added, subtracted and multiplied polynomials. They can also be divided, which is useful in factorising polynomials.

Before looking at long division of polynomials, first consider long division of numbers.

$$\begin{array}{r}
26 \\
9\overline{)237} \\
\underline{18} \\
57 \\
\underline{54} \\
3
\end{array}$$

1 The working for using long division to find the result to 237 ÷ 9 is shown at right.

- **a** Explain how 2 is obtained in the top line.
- **b** Explain why 18 is written below 23.
- **c** What is the result of 23 − 18? Identify where this is written in the working.
- **d** How is the new number to divide into (57) formed?
- **e** How many times does 9 go into 57? Where is this written?
- **f** Where does 54 come from?
- **g** How is 3 obtained?

2 In long division, when the **dividend** is divided by the **divisor**, the **quotient** is written on the top line and the **remainder** is in the last line. Identify the dividend, divisor, quotient and remainder for 237 ÷ 9.

3 Copy and complete these statements.

- **a** 237 ÷ 9 = _____ remainder _____
- **b** 237 = 9 × _____ + 3

KEY IDEAS

- Long division is used to divide polynomials. At each stage, the leading term of the dividend is divided by the leading term of the divisor.

$$\begin{array}{rrl}
 & x + 3 & \leftarrow \text{quotient} \\
\text{divisor} \rightarrow x + 2 & \overline{)x^2 + 5x + 8} & \leftarrow \text{dividend} \\
 & \underline{x^2 + 2x\phantom{{}+8}} & \\
 & 3x + 8 & \\
 & \underline{3x + 6} & \\
 & 2 & \leftarrow \text{remainder}
\end{array}$$

- The quotient and remainder are important elements to identify. In the example, the quotient is $x + 3$ and the remainder is 2. This means that:
  $(x^2 + 5x + 8) \div (x + 2) = x + 3$ remainder 2
  or $\dfrac{x^2 + 5x + 8}{x + 2} = x + 3 + \dfrac{2}{x + 2}$.
- Alternatively, this can be written as $x^2 + 5x + 8 = (x + 2)(x + 3) + 2$; that is:
  dividend = divisor × quotient + remainder
- In general, if $P(x)$ is divided by $(x - a)$ to produce the quotient $Q(x)$ and the remainder R, then $P(x) = (x - a)Q(x) + R$.

EXERCISE 6B Division of polynomials

UNDERSTANDING AND FLUENCY

1 Copy and complete each statement.

- **a** If $152 \div 8 = 19$, then $152 = 8 \times$ ___.
- **b** If $9475 \div 25 = 379$, then $9475 =$ ___ $\times$ ___.
- **c** If $321 \div 7 = 45$ remainder 6, then $351 = 7 \times 45 +$ ___.
- **d** If $2386 \div 16 = 149$ remainder 2, then $2386 = 16 \times$ ___ $+$ ___.

2 Copy and complete each statement.

- **a** If $(x^2 + 5x + 6) \div (x + 2) = x + 3$,
  then $x^2 + 5x + 6 = (x + 2)($____$)$.
- **b** If $(x^3 + 2x^2 + 3x + 2) \div (x + 1) = x^2 + x + 2$,
  then $x^3 + 2x^2 + 3x + 2 = ($____$)($____$)$.
- **c** If $(2x^2 - 3x - 4) \div (x - 3) = 2x + 3$ remainder 5,
  then $2x^2 - 3x - 4 = (x - 3)($____$) + 5$.
- **d** If $(x^3 + 4x^2 - x + 3) \div (x - 1) = x^2 + 5x + 4$ remainder 7,
  then $x^3 + 4x^2 - x + 3 = ($____$)($____$) +$ ____.

EXAMPLE 6B-1 Dividing a quadratic polynomial by a linear expression

Use long division to find the quotient and remainder for $(2x^2 + 5x - 1) \div (x + 4)$.

THINK	WRITE
	Note: each stage of working has been shown separately.
1 Divide the leading term in the dividend ($2x^2$) by the leading term of the divisor (x). $2x^2 \div x = 2x$. Write the result of $2x$ above $5x$ in the quotient line. Remember to align like terms in columns.	$\begin{array}{r} 2x \\ x+4\,\big)\overline{2x^2+5x-1} \end{array}$
2 Work out the remainder by first multiplying $2x$ by the divisor. $2x(x + 4) = 2x^2 + 8x$. Write the result underneath, then subtract like terms.	$\begin{array}{lr} & 2x \\ & x+4\,\big)\overline{2x^2+5x-1} \\ 2x(x+4) & \underline{2x^2+8x} \\ & -3x-1 \end{array}$
3 Divide the leading term of $-3x - 1$ by the leading term of the divisor. $-3x \div x = -3$. Write the result of -3 above -1 in the quotient line.	
4 Work out the remainder by first multiplying -3 by the divisor. $-3(x + 4) = -3x - 12$. Write the result underneath, then subtract like terms.	$\begin{array}{lr} & 2x-3 \\ & x+4\,\big)\overline{2x^2+5x-1} \\ 2x(x+4) & \underline{2x^2+8x} \\ & -3x-1 \\ -3(x+4) & \underline{-3x-12} \\ & 11 \end{array}$
5 Identify the quotient and the remainder.	Quotient is $2x - 3$, remainder is 11.

10A

UNDERSTANDING AND FLUENCY

3 Copy and complete the working for each long division problem.

a
$$\begin{array}{r} x + \\ x+2\,\overline{)\,x^2+6x+1} \\ \underline{x^2 + } \\ +1 \\ \underline{4x + } \\ -7 \end{array}$$

Quotient is ______, remainder is ____.

b
$$\begin{array}{r} 4x - \\ x-1\,\overline{)\,4x^2-7x+5} \\ \underline{4x^2 - } \\ -3x + \\ \underline{+3} \\ 2 \end{array}$$

Quotient is ______, remainder is ____.

4 Use long division to find the quotient and remainder for each division problem.

a $(x^2 + 7x + 8) \div (x + 3)$

b $(x^2 + 3x - 1) \div (x - 2)$

c $(x^2 + 2x + 3) \div (x + 4)$

d $(x^2 - 5x - 4) \div (x + 2)$

e $(2x^2 - 3x - 11) \div (x - 3)$

f $(3x^2 + x + 2) \div (x - 1)$

5 Write your answers to question **4** in the form: dividend = divisor × quotient + remainder. For example, $2x^2 + 5x - 1 = (x + 4)(2x - 3) + 11$.

6 Expand and simplify the right side of each statement found in question **5** to verify that the statement is true.

7 **a** Divide $x^2 - 2x - 24$ by $(x + 4)$ to find the quotient and remainder.

b Write the division problem in the form:
dividend = divisor × quotient + remainder.

c What does the value of the remainder tell you about the divisor?

EXAMPLE 6B-2 Dividing a polynomial by a linear expression

Use long division to find the quotient and remainder for $(x^3 + 2x^2 - 9x - 3) \div (x - 2)$.

THINK	WRITE
1 Divide leading term in dividend (x^3) by leading term of divisor (x) and write the result of x^2 in the quotient line.	
2 Expand $x^2(x - 2)$ to give $x^3 - 2x^2$ and write the result underneath. Subtract like terms.	$\begin{array}{lr} & x^2+4x-1 \\ & x-2\,\overline{)\,x^3+2x^2-9x-3} \\ x^2(x-2) & \underline{x^3-2x^2} \\ & 4x^2-9x-3 \\ 4x(x-2) & \underline{4x^2-8x} \\ & -x-3 \\ -1(x-2) & \underline{-x+2} \\ & -5 \end{array}$
3 Divide $4x^2$ by x and write the result of $4x$ in the quotient line.	
4 Expand $4x(x - 2)$ to give $4x^2 - 8x$ and write the result underneath. Subtract like terms.	
5 Divide $-x$ by x and write the result of -1 in the quotient line.	
6 Expand $-1(x - 2)$ to give $-x + 2$ and write the result underneath. Subtract like terms.	
7 Identify the quotient and the remainder.	Quotient is $x^2 + 4x - 1$, remainder is -5.

8 Use long division to find the quotient and remainder for each division problem.

- **a** $(x^3 - 2x^2 - 5x + 7) \div (x - 3)$
- **b** $(x^3 + 3x^2 + 7x - 1) \div (x + 1)$
- **c** $(x^3 + 6x^2 - 4x - 15) \div (x - 2)$
- **d** $(x^3 - 3x^2 - 20x + 25) \div (x + 4)$
- **e** $(2x^3 + 3x^2 - 5x - 4) \div (x + 2)$
- **f** $(3x^3 - 2x^2 - 7x + 2) \div (x - 1)$
- **g** $(5x^3 - 12x^2 - 6x - 15) \div (x - 3)$
- **h** $(4x^3 + 9x^2 - 8x - 1) \div (x + 1)$
- **i** $(x^4 + 2x^3 - 4x^2 - 2x + 8) \div (x + 1)$
- **j** $(x^4 - 5x^3 + 2x^2 - x - 3) \div (x - 2)$

EXAMPLE 6B-3 Dividing a polynomial that has coefficients of zero by a linear expression

Use long division to find the quotient and remainder for $(3x^4 + 2x^2 - 5) \div (x + 1)$.

THINK

1. Write the 'missing' terms in the dividend with zero as the coefficient to keep like terms in columns.
2. Divide $3x^4$ by x and write $3x^3$ in quotient line.
3. Expand $3x^3(x + 1)$ to give $3x^4 + 3x^3$. Subtract like terms.
4. Divide $-3x^3$ by x and write $-3x^2$ in quotient line.
5. Expand $-3x^2(x + 1)$ to give $-3x^3 - 3x^2$. Subtract like terms.
6. Divide $5x^2$ by x and write $5x$ in quotient line.
7. Expand $5x(x + 1)$ to give $5x^2 + 5x$. Subtract like terms.
8. Divide $-5x$ by x and write -5 in quotient line.
9. Expand $-5(x + 1)$ to give $-5x - 5$. Subtract like terms.
10. Identify the quotient and the remainder.

WRITE

$$
\begin{array}{ll}
 & 3x^3 - 3x^2 + 5x - 5 \\
 & x + 1\,\overline{)\,3x^4 + 0x^3 + 2x^2 + 0x - 5} \\
3x^3(x + 1) & \underline{3x^4 + 3x^3} \\
 & \qquad\qquad -3x^3 + 2x^2 + 0x - 5 \\
-3x^2(x + 1) & \qquad\qquad \underline{-3x^3 - 3x^2} \\
 & \qquad\qquad\qquad\quad 5x^2 + 0x - 5 \\
5x(x + 1) & \qquad\qquad\qquad\quad \underline{5x^2 + 5x} \\
 & \qquad\qquad\qquad\qquad\quad -5x - 5 \\
-5(x + 1) & \qquad\qquad\qquad\qquad\quad \underline{-5x - 5} \\
 & \qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Quotient is $3x^3 - 3x^2 + 5x - 5$, remainder is 0.

9 Use long division to find the quotient and remainder for each division problem.

- **a** $(x^3 + x + 21) \div (x + 3)$
- **b** $(2x^3 - 3x^2 - 6) \div (x - 2)$
- **c** $(4x^4 - 3x^2 - 5) \div (x - 1)$
- **d** $(3x^4 - 41) \div (x + 2)$

10 **a** If $P(x) = x^3 - 3x^2 - 10x + k$, for what value of k does $P(x) \div (x - 2)$ give a remainder of zero.

b Use the divisor and quotient to write the linear factor and the quadratic factor of $P(x)$. Hence, show that you can write $P(x)$ as the product of three linear factors.

11 **a** If $P(x) = x^3 + 9x^2 + 23x + 15$, for what three values of k does $P(x) \div (x + k)$ give a remainder of zero.

b Write $P(x)$ as the product of three linear factors.

CHALLENGE

12 Use long division to find the quotient and remainder for each division problem.

- **a** $(x^3 - 3x^2 + 2x - 4) \div (x^2 + 1)$
- **b** $(x^4 + x^3 - 7x^2 + 3x + 5) \div (x^2 - 3)$
- **c** $(2x^4 + 6x^2 - 1) \div (x^2 + 2)$

Reflect

Why is knowing how to divide polynomials useful?

6C Remainder and factor theorems

Start thinking!

Consider the polynomial $P(x) = x^3 + x^2 - 10x + 8$.

1 a Use long division to find the remainder when $P(x)$ is divided by $(x - 3)$.
  b Evaluate $P(3)$. What do you notice?

2 a Use long division to find the remainder when $P(x)$ is divided by $(x + 2)$.
  b Evaluate $P(-2)$. What do you notice?

3 a Without using long division, use the pattern you noticed in questions **1** and **2** to show how to work out the remainder when $P(x)$ is divided by $(x - 4)$.
  b Use long division to verify your answer to part **a**.

4 a Evaluate $P(1)$ to find the remainder when $P(x)$ is divided by $(x - 1)$.
  b Use long division to verify your answer to part **a**.

5 The pattern or shortcut you have used is called the **remainder theorem**. Explain how you can find the remainder without using long division when $P(x)$ is divided by:
  a $x - 5$ b $x + 1$ c $x - 2$

6 What does the remainder in question **4** tell you about the relationship between $P(x)$ and $x - 1$? This is the basis for the **factor theorem**.

KEY IDEAS

- Remainder theorem
  - When a polynomial $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$. For example:
    when $P(x)$ is divided by $(x - 2)$, the remainder is $P(2)$
    when $P(x)$ is divided by $(x + 3)$, the remainder is $P(-3)$.
- Factor theorem
  - When a polynomial $P(x)$ is divided by $(x - a)$ and the remainder $P(a)$ is zero, then $(x - a)$ is a factor of $P(x)$. For example, $P(x) = x^3 + x^2 - 10x + 8$ has a factor of $(x - 1)$, as $P(1) = 0$.
- If a linear factor of a cubic polynomial is known, long division can be used to find the quadratic factor. For example, dividing $P(x) = x^3 + x^2 - 10x + 8$ by $(x - 1)$ gives a quotient of $x^2 + 2x - 8$.
- Factorising this quotient allows us to write the polynomial as a product of its linear factors.
- In general, if $P(x)$ is divided by a factor $(x - a)$ to produce the quotient $Q(x)$, then $P(x) = (x - a)Q(x)$.

$$\begin{aligned} P(x) &= x^3 + x^2 - 10x + 8 \\ &= (x - 1)(x^2 + 2x - 8) \\ &= (x - 1)(x - 2)(x + 4) \end{aligned}$$

EXERCISE 6C Remainder and factor theorems

EXAMPLE 6C-1 Using the remainder theorem

Find the remainder when $x^3 + 4x^2 + x - 6$ is divided by: **a** $x - 2$ **b** $x + 3$

THINK	WRITE
Name the polynomial.	Let $P(x) = x^3 + 4x^2 + x - 6$.
a For divisor of $(x - 2)$, calculate $P(2)$ and hence write the remainder.	**a** $P(2) = 8 + 16 + 2 - 6 = 20$ Remainder is 20.
b For divisor of $(x + 3)$, calculate $P(-3)$ and hence write the remainder.	**b** $P(-3) = -27 + 36 - 3 - 6 = 0$ Remainder is 0.

1 Find the remainder when $x^3 - 5x^2 - 8x + 12$ is divided by:

a $x - 1$ **b** $x + 1$ **c** $x - 2$ **d** $x + 2$.

2 Find the remainder when $2x^3 + 3x^2 - 8x + 3$ is divided by:

a $x + 1$ **b** $x - 2$ **c** $x + 2$ **d** $x + 3$.

3 Find the remainder when $x^4 - 7x^3 + 5x^2 + 31x - 30$ is divided by:

a $x - 2$ **b** $x + 1$ **c** $x - 3$ **d** $x + 4$.

EXAMPLE 6C-2 Deciding if the divisor is a factor of the polynomial

Decide if each divisor is a factor of $x^3 + 2x^2 - x - 2$. **a** $x - 3$ **b** $x + 1$

THINK	WRITE
Name the polynomial.	Let $P(x) = x^3 + 2x^2 - x - 2$.
a For $(x - 3)$ to be a factor, the remainder should be zero. Check if $P(3) = 0$.	**a** $P(3) = 27 + 18 - 3 - 2 = 40$ $(x - 3)$ is not a factor of $P(x)$.
b For $(x + 1)$ to be a factor, the remainder should be zero. Check if $P(-1) = 0$.	**b** $P(-1) = -1 + 2 + 1 - 2 = 0$ $(x + 1)$ is a factor of $P(x)$.

4 Explain how you know if the divisor is a factor of the polynomial.

5 Decide if each divisor is a factor of $x^3 + 3x^2 - 6x - 8$.

a $x - 1$ **b** $x + 1$ **c** $x - 3$ **d** $x + 3$
e $x - 4$ **f** $x + 4$ **g** $x + 2$ **h** $x - 2$

6 Decide if each divisor is a factor of $x^3 - 4x^2 - 7x + 10$.

a $x - 2$ **b** $x - 3$ **c** $x - 1$ **d** $x + 3$
e $x + 1$ **f** $x + 2$ **g** $x - 5$ **h** $x - 4$

EXAMPLE 6C-3 Using the factor theorem to find a linear factor of a polynomial

Use the factor theorem to find a linear factor of the polynomial $P(x) = x^3 + 5x^2 - 2x - 24$.

THINK

1 Write the polynomial.

2 Look for a value of x where $P(x) = 0$. Try $P(1)$, $P(-1)$, $P(2)$, etc. That is, try x values that are factors of the constant term in $P(x)$.

3 Use the factor theorem to identify a linear factor of $P(x)$.

WRITE

$P(x) = x^3 + 5x^2 - 2x - 24$

$P(1) = 1 + 5 - 2 - 24 = -20 \neq 0$
$P(-1) = -1 + 5 + 2 - 24 = -18 \neq 0$
$P(2) = 8 + 20 - 4 - 24 = 0$

So $(x - 2)$ is a factor of $P(x)$.

7 Use the factor theorem to find a linear factor of each polynomial.

a $P(x) = x^3 + 8x^2 + 9x - 18$
b $P(x) = x^3 - x^2 - 14x + 24$
c $P(x) = x^3 - 4x^2 - 9x + 36$
d $P(x) = x^3 - 19x - 30$
e $P(x) = 3x^3 - 10x^2 + x + 6$
f $P(x) = 2x^3 - 5x^2 - 14x + 8$
g $P(x) = 8x^3 - 26x^2 + 17x + 6$
h $P(x) = 4x^3 + 4x^2 - 21x + 9$

8 Fully factorise each expression.

a $(x - 6)(x^2 + 7x + 12)$
b $(x + 4)(x^2 - 9x + 14)$
c $(x + 3)(x^2 - x - 30)$
d $(x - 2)(x^2 - 16)$
e $(x + 1)(x^2 + 4x + 4)$
f $(x - 7)(x^2 + 5x - 24)$

EXAMPLE 6C-4 Factorising a cubic polynomial

Factorise $x^3 - 7x^2 + 7x + 15$.

THINK

1 Name the polynomial.

2 Look for a value of x where $P(x) = 0$. Try $P(1)$, $P(-1)$, $P(2)$, … (factors of 15).

3 Identify a linear factor of $P(x)$.

4 Use long division to find the quotient when $P(x)$ is divided by $(x + 1)$. You will know you have performed the division correctly as the remainder should be zero.

5 Write $P(x)$ as the product of the divisor and quotient.

6 Factorise the quadratic factor.

WRITE

Let $P(x) = x^3 - 7x^2 + 7x + 15$.

$P(1) = 1 - 7 + 7 + 15 \neq 0$
$P(-1) = -1 - 7 - 7 + 15 = 0$

So $(x + 1)$ is a factor of $P(x)$.

$$
\begin{array}{r}
x^2 + 8x + 15 \\
x + 1\,\overline{)\,x^3 - 7x^2 + 7x + 15} \\
-(x^3 + x^2) \\
\hline
-8x^2 + 7x + 15 \\
-(-8x^2 - 8x) \\
\hline
15x + 15 \\
-(15x + 15) \\
\hline
0
\end{array}
$$

$P(x) = (x + 1)(x^2 - 8x + 15)$
$\quad = (x + 1)(x - 3)(x - 5)$

PROBLEM SOLVING AND REASONING

9 Factorise each polynomial.

a $x^3 + 6x^2 + 3x - 10$
b $x^3 + 3x^2 - 4x - 12$
c $x^3 + 4x^2 - 19x + 14$
d $x^3 + 5x^2 - 4x - 20$
e $x^3 - 5x^2 - 8x + 48$
f $x^3 - 2x^2 - 9x + 18$
g $2x^3 + x^2 - 8x - 4$
h $4x^3 - 9x^2 - 19x + 30$
i $6x^3 - 23x^2 - 6x + 8$
j $12x^3 - 17x^2 + 2x + 3$

10 Factorise each polynomial in question 7.

11 What is the maximum number of factors $P(x)$ can have if $P(x)$ is:

a a quadratic polynomial?
b cubic polynomial?
c quartic polynomial?
d polynomial of degree 7?
e polynomial of degree 12?
f polynomial of degree n?

12 a How many linear factors does the polynomial $P(x)$ have if $P(x) = (x + 2)(x - 3)(x + 4)$?

b Without expanding, what is the constant term of $P(x)$? Explain.

c To find linear factors using the factor theorem, you look for values of x that make $P(x) = 0$. What are these x values and how do they relate to the constant term?

13 Consider the polynomial $P(x) = x^3 + 2x^2 - 5x - 6$.

a Use the factor theorem to find a linear factor of $P(x)$.

b Use the factor theorem to find another linear factor of $P(x)$.

c Use the factor theorem to find a third linear factor of $P(x)$.

d Write $P(x)$ as a product of three factors.

e Check your answer to part d by expanding and simplifying the product.

NOTE Remember to try factors of −6 to identify the x value that makes $P(x) = 0$.

14 a Use your answers to question 5 to identify three factors of the polynomial $P(x)$.

b Write $P(x)$ as a product of three factors.

c Check your answer to part b by expanding and simplifying the product.

15 a Use your answers to question 6 to identify three factors of the polynomial $P(x)$.

b Write $P(x)$ as a product of three factors.

16 Using only the factor theorem, find three factors of $x^3 + 2x^2 - 11x - 12$ and hence write the polynomial as a product of three factors.

CHALLENGE

17 Explain why the strategy used in question 16 is not suitable to factorise each of these polynomials.

a $6x^3 + 13x^2 + 4x - 3$
b $x^3 + 5x^2 + 10x + 8$

18 Find a polynomial of degree 3 with leading coefficient of 1 that has a remainder of 5 when divided by $x - 2$.

19 Factorise each quartic polynomial.

a $x^4 - x^3 - 7x^2 + x + 6$
b $x^4 + 8x^3 + 17x^2 - 2x - 24$
c $2x^4 + 13x^3 + 21x^2 + 2x - 8$

Reflect

How does the factor theorem help you factorise a polynomial?

6D Solving polynomial equations

Start thinking!

1 Consider the quadratic polynomial equation $x^2 + 3x - 18 = 0$.

a What is the maximum number of solutions this equation can have?

b Check which of these x values is a solution to the equation.

i $x = -1$ **ii** $x = 1$ **iii** $x = -2$ **iv** $x = 3$

v $x = -4$ **vi** $x = 5$ **vii** $x = -6$ **viii** $x = 7$

c Without substitution, how could you identify if each x value is a *possible* solution? (Hint: look at the constant term on the left side of the equation.)

2 Consider the quadratic polynomial equation $x^2 + x - 6 = 0$.

a Without substitution, identify which of these x values is a *possible* solution to the equation? Explain your decision.

i $x = -1$ **ii** $x = 5$ **iii** $x = -3$ **iv** $x = 4$

v $x = -2$ **vi** $x = 6$ **vii** $x = 2$ **viii** $x = 8$

b From the set of possible x values identified in part **a**, check which ones are a solution.

3 Repeat question **2** for the polynomial equation $x^3 + x^2 - 4x - 4 = 0$.

4 **a** Factorise the left side of the equation in question **1**.

b Explain how you can use the Null Factor Law to solve the equation.

5 Solve the equation in question **2** using the Null Factor Law.

KEY IDEAS

- Null Factor Law: if $a \times b = 0$ then $a = 0$ or $b = 0$ or both a and b are 0.
- The polynomial equation $P(x) = 0$ can be solved by applying the Null Factor Law. $P(x)$ must be in factor form.
- If $P(x) = (x - a)(x - b)(x - c)$ then $P(x) = 0$ has the solution $x = a$, $x = b$ or $x = c$.
- To factorise $P(x)$ so it is a product of linear factors:
  1. apply the factor theorem to find a linear factor
  2. divide $P(x)$ by the linear factor to obtain the quotient
  3. factorise the quotient.
- If $P(x)$ is a cubic polynomial, the quotient will be a quadratic factor that may be factorised into two linear factors.
- If $P(x)$ is a quartic polynomial, the quotient will be a cubic factor. The strategy of using the factor theorem and long division will need to be repeated with the quotient.

EXERCISE 6D Solving polynomial equations

1 Factorise the left side of each quadratic equation.

a $x^2 + 5x + 4 = 0$
b $x^2 - 9x + 18 = 0$
c $x^2 - 2x - 8 = 0$
d $x^2 + 6x + 9 = 0$
e $x^2 - 25 = 0$
f $2x^2 - 14x = 0$

2 Solve each equation in question 1 using the Null Factor Law.

EXAMPLE 6D-1 Solving polynomial equations in factor form

Solve each equation. a $(x - 1)(x + 3)(x - 2) = 0$ b $(2x + 1)(x - 5)(x - 4)(x + 4) = 0$

THINK	WRITE
a 1 As the left side (LS) is in factor form and the right side (RS) is 0, apply the Null Factor Law.	a $(x - 1)(x + 3)(x - 2) = 0$ $x - 1 = 0$ or $x + 3 = 0$ or $x - 2 = 0$
2 Solve each linear equation.	$x = 1$ or $x = -3$ or $x = 2$
3 Write the solution.	$x = -3, 1$ or 2
b 1 Apply the Null Factor Law.	b $(2x + 1)(x - 5)(x - 4)(x + 4) = 0$ $2x + 1 = 0$ or $x - 5 = 0$ or $x - 4 = 0$ or $x + 4 = 0$
2 Solve each linear equation.	$x = -\frac{1}{2}$ or $x = 5$ or $x = 4$ or $x = -4$
3 Write the solution.	$x = -4, -\frac{1}{2}, 4$ or 5

UNDERSTANDING AND FLUENCY

3 Solve each equation.

a $(x + 2)(x + 5)(x - 4) = 0$
b $(x + 1)(x - 3)(x + 4) = 0$
c $(x - 6)(x - 2)(x + 3) = 0$
d $x(x + 2)(x - 9) = 0$
e $(2x - 1)(x + 1)(x - 1) = 0$
f $(3x - 2)(2x + 7)(x + 5) = 0$

4 Solve each equation.

a $(x + 3)(x - 4)(x + 7)(x - 1) = 0$
b $(x - 2)(x - 5)(x - 3)(x + 6) = 0$
c $(x + 4)(x + 1)(x + 2)^2 = 0$
d $x^2(x + 5)(x - 6) = 0$
e $5x(3x + 1)(x + 4)(4 - x) = 0$
f $(4x - 3)(2x + 5)(x^2 + 1) = 0$

5 Fully factorise the left side of each equation. Hence, solve each equation.

a $(x + 1)(x^2 + 8x + 12) = 0$
b $(x - 3)(x^2 - x - 20) = 0$
c $(x - 2)(x^2 - 6x + 8) = 0$
d $(x + 4)(x^2 + 6x - 7) = 0$
e $(x + 3)(x^2 - 16) = 0$
f $x(x - 1)(x^2 - 1) = 0$
g $(x - 5)(x^2 - 10x + 25) = 0$
h $(x + 2)(x^2 + 6x + 9) = 0$
i $(x - 4)(x^2 - x - 6) = 0$
j $(x + 1)(x^2 + 2x + 2) = 0$
k $3(x + 3)(5x^2 + 30x) = 0$
l $(x - 2)(2x^2 + 4x - 6) = 0$

EXAMPLE 6D-2 Solving a cubic polynomial equation

Solve $x^3 - 7x^2 + 7x + 15 = 0$.

THINK	WRITE
1 Name the polynomial.	Let $P(x) = x^3 - 7x^2 + 7x + 15$.
2 Use the factor theorem to identify a linear factor of $P(x)$.	$P(1) = 1 - 7 + 7 + 15 \neq 0$ $P(-1) = -1 - 7 - 7 + 15 = 0$ So $(x + 1)$ is a factor of $P(x)$.
3 Use long division to find the quotient when $P(x)$ is divided by $(x + 1)$.	$\begin{array}{r l} & x^2 - 8x + 15 \\ x+1 \,\big) & \overline{x^3 - 7x^2 + 7x + 15} \\ & \underline{x^3 + x^2} \\ & -8x^2 + 7x + 15 \\ & \underline{-8x^2 - 8x} \\ & 15x + 15 \\ & \underline{15x + 15} \\ & 0 \end{array}$
4 Write $P(x)$ as the product of the divisor and quotient.	$P(x) = (x + 1)(x^2 - 8x + 15)$
5 Factorise the quadratic factor.	$= (x + 1)(x - 3)(x - 5)$
6 Solve $P(x) = 0$ using the Null Factor Law.	For $P(x) = 0$, $(x + 1)(x - 3)(x - 5) = 0$ $x = -1, 3$ or 5

UNDERSTANDING AND FLUENCY

6 Solve each equation.

a $x^3 + 6x^2 + 5x - 12 = 0$
b $x^3 - 5x^2 - 4x + 20 = 0$
c $x^3 + x^2 - 36x - 36 = 0$
d $x^3 + 10x^2 + 21x = 0$
e $x^3 - 7x - 6 = 0$
f $x^3 + 2x^2 + 5x + 10 = 0$
g $x^3 + 3x^2 - 9x - 27 = 0$
h $x^3 + 9x^2 + 24x + 16 = 0$
i $x^3 - 3x^2 - 3x - 4 = 0$
j $x^3 - 6x^2 + 12x - 8 = 0$

7 Solve each equation by first taking out a common factor.

a $2x^3 - 2x^2 - 20x - 16 = 0$
b $3x^3 - 15x^2 - 3x + 15 = 0$
c $5x^3 + 10x^2 - 20x - 40 = 0$
d $4x^3 + 24x^2 + 44x + 24 = 0$

8 Solve each equation by first taking out a negative common factor.

a $-x^3 - 2x^2 + 9x + 18 = 0$
b $-x^3 + 4x^2 + 17x - 60 = 0$
c $-2x^3 + 8x^2 + 2x - 8 = 0$
d $-3x^3 - 3x^2 + 24x + 36 = 0$

9 Solve each equation using the quadratic formula. Write the solutions as exact values.

a $x^3 + x^2 - 3x + 1 = 0$
b $x^3 + 6x^2 + 9x + 2 = 0$
c $x^3 + x^2 - 10x - 12 = 0$
d $x^3 + 4x^2 - 27x - 20 = 0$

UNDERSTANDING AND FLUENCY

10 Follow these steps to solve $x^4 + x^3 - 7x^2 - x + 6 = 0$.

- **a** Name the polynomial on the left side of the equation as $P(x)$ and identify one of its linear factors.
- **b** Use long division to find the quotient when $P(x)$ is divided by this linear factor. Write $P(x)$ as the product of the divisor and quotient, $Q(x)$.
- **c** Identify a linear factor of $Q(x)$ and use long division to find its quadratic factor.
- **d** Factorise the quadratic factor.
- **e** Write $P(x)$ as a product of four linear factors and hence solve $P(x) = 0$ using the Null Factor Law.

11 Solve each equation.

- **a** $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$
- **b** $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$
- **c** $x^4 + 7x^3 + 8x^2 - 28x - 48 = 0$
- **d** $x^4 - x^3 - 19x^2 - 11x + 30 = 0$
- **e** $x^4 - 5x^3 + 20x - 16 = 0$
- **f** $x^4 - 13x^2 + 36 = 0$

PROBLEM SOLVING AND REASONING

12 What is the maximum number of solutions to $P(x) = 0$ if $P(x)$ is:

- **a** a linear polynomial?
- **b** a quadratic polynomial?
- **c** a cubic polynomial?
- **d** a quartic polynomial?
- **e** a polynomial of degree 6?
- **f** a polynomial of degree n?

13 **a** Using only the factor theorem, find three linear factors of $P(x)$ where $P(x) = x^3 - 2x^2 - 5x + 6$.

- **b** Hence, solve $P(x) = 0$.

14 Explain why $x^3 + x^2 - 2x - 8 = 0$ has only one solution.

15 Explain why $x^3 - 5x^2 + 3x + 9 = 0$ has only two solutions.

16 The volume of a vanilla slice is 192 cm^3. Its length is twice the height and the width is 2 cm more than the height.

- **a** Write a polynomial to represent the volume of the slice.
- **b** Solve a polynomial equation to find the dimensions of the slice.

17 The volume of a Toblerone chocolate box is 450 cm^3. The height of its triangular face is 1 cm less than the base and its perpendicular length is five times the size of the base. Find its dimensions.

CHALLENGE

18 $P(x) = x^4 - 2x^3 - 13x^2 + 14x + 24$ has the quadratic factor $(x^2 - x - 2)$. Factorise $P(x)$ and hence solve $P(x) = 0$.

19 Solve each equation.

- **a** $x^5 - x^4 - 17x^3 - 19x^2 + 16x + 20 = 0$
- **b** $x^5 - 2x^4 - 15x^3 + 20x^2 + 44x - 48 = 0$

Reflect

In what form does a polynomial equation need to be before the Null Factor Law can be used?

6E Graphs of polynomial relationships

Start thinking!

A polynomial relationship can be shown as a graph. In chapters 4 and 5, you worked with linear and quadratic relationships. Here you will look at cubic and quartic relationships.

1 The basic cubic graph has the rule $y = x^3$.

- **a** Draw the graph using a table of values (for $-3 \le x \le 3$) or digital technology.
- **b** Identify the x- and y-intercepts.
- **c** An important feature on this graph is the **point of inflection** at (0, 0). Mark this on your graph.

2 The basic quartic graph has the rule $y = x^4$.

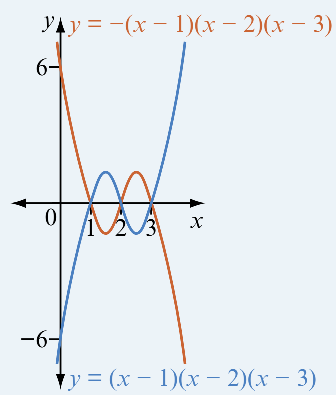
- **a** Draw the graph using a table of values (for $-3 \le x \le 3$) or digital technology.
- **b** Identify the x- and y-intercepts.
- **c** This graph looks similar to $y = x^2$. How is it different? Draw the graph of $y = x^2$ on the same Cartesian plane to illustrate your answer.

3 Sketch the graph of each polynomial relationship. To give a sense of scale, you can show the coordinates of a point on the graph. Label the point where $x = 2$.

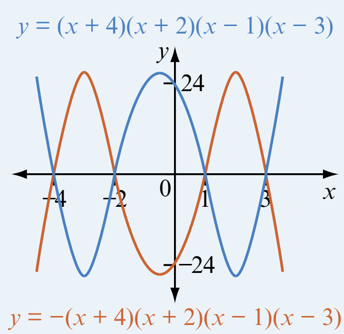
a $y = -x^3$ **b** $y = -x^4$

KEY IDEAS

- To sketch a polynomial relationship:
  1. write the polynomial in factor form
  2. identify the x-intercepts (find x when $y = 0$)
  3. identify the y-intercept (find y when $x = 0$)
  4. draw a smooth curve through the known points
  5. if necessary, find the coordinates of another point to confirm the orientation of the graph.



- Graphs of cubic relationships
  - ▷ The graph of $y = (x-a)(x-b)(x-c)$ has x-intercepts a, b and c, and y-intercept $-abc$. The curve starts from the bottom left of the Cartesian plane.
  - ▷ The graph of $y = -(x-a)(x-b)(x-c)$ is the reflection in the x-axis of $y = (x-a)(x-b)(x-c)$.

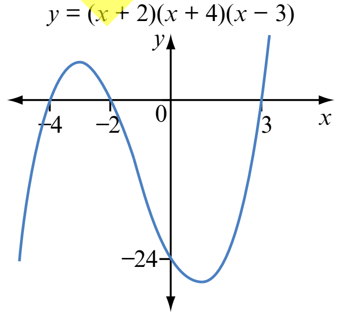


- Graphs of quartic relationships
  - ▷ The graph of $y = (x-a)(x-b)(x-c)(x-d)$ has x-intercepts a, b, c and d, and y-intercept $abcd$. The curve starts from the top left of the Cartesian plane.
  - ▷ The graph of $y = -(x-a)(x-b)(x-c)(x-d)$ is the reflection in the x-axis of $y = (x-a)(x-b)(x-c)(x-d)$.

EXERCISE 6E Graphs of polynomial relationships

EXAMPLE 6E-1 Sketching a cubic relationship using intercepts

Sketch the graph of $y = (x + 2)(x + 4)(x - 3)$.

THINK	WRITE
1 Substitute $y = 0$ and use the Null Factor Law to find the x-intercepts.	$y = (x + 2)(x + 4)(x - 3)$ When $y = 0$, $(x + 2)(x + 4)(x - 3) = 0$ $x + 2 = 0$ or $x + 4 = 0$ or $x - 3 = 0$ $x = -2, -4$ or 3 x-intercepts are -4, -2 and 3.
2 Substitute $x = 0$ to find the y-intercept.	When $x = 0$, $y = (2)(4)(-3) = -24$ y-intercept is -24.
3 Mark the four intercepts on a Cartesian plane and draw a smooth curve through them. Curve starts from bottom left of Cartesian plane. Label with the rule.	

UNDERSTANDING AND FLUENCY

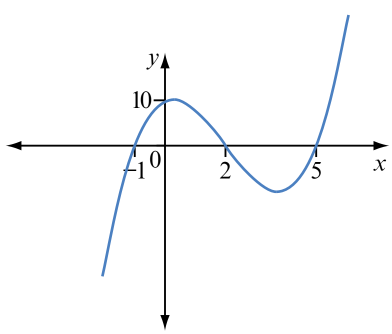
1 Sketch the graph of each cubic relationship.

a $y = (x + 1)(x + 3)(x - 1)$
b $y = (x - 2)(x - 5)(x + 1)$
c $y = (x - 4)(x + 2)(x + 3)$
d $y = (x - 3)(x - 1)(x - 5)$
e $y = x(x + 3)(x - 3)$
f $y = 2x(x + 4)(x - 1)$

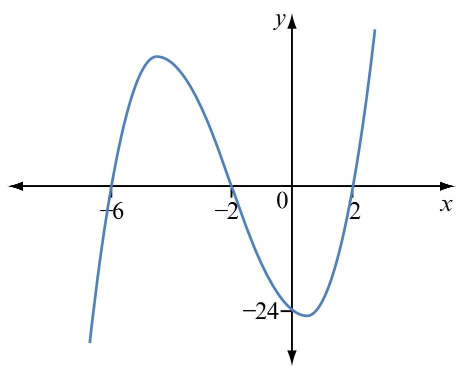
2 For each cubic relationship:

i identify the x-intercepts
ii identify the y-intercept
iii write the rule.

a



b



3 **a** Find the x- and y-intercepts for:

  i $y = (x + 7)(x - 2)(x - 3)$ **ii** $y = -(x + 7)(x - 2)(x - 3)$

  b How is the graph of $y = -(x + 7)(x - 2)(x - 3)$ different from the graph of $y = (x + 7)(x - 2)(x - 3)$?

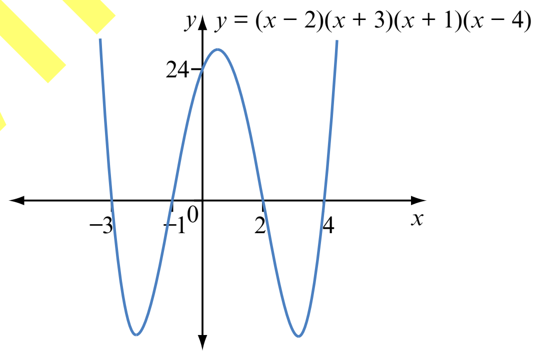
4 Sketch the graph of each cubic relationship.

  a $y = -(x + 5)(x - 1)(x - 3)$ **b** $y = -(x - 6)(x + 1)(x - 2)$

  c $y = -(x + 2)(x + 3)(x - 5)$ **d** $y = -x(x + 7)(x - 4)$

EXAMPLE 6E-2 Sketching a quartic relationship using intercepts

Sketch the graph of $y = (x - 2)(x + 3)(x + 1)(x - 4)$.

THINK	WRITE
1 Substitute $y = 0$ and use the Null Factor Law to find the x-intercepts.	$y = (x - 2)(x + 3)(x + 1)(x - 4)$ When $y = 0$, $(x - 2)(x + 3)(x + 1)(x - 4) = 0$ $x = 2$ or $x = -3$ or $x = -1$ or $x = 4$ x-intercepts are -3, -1, 2 and 4.
2 Substitute $x = 0$ to find the y-intercept.	When $x = 0$, $y = (-2)(3)(1)(-4) = 24$ y-intercept is 24.
3 Mark the five intercepts on a Cartesian plane and draw a smooth curve through them. Curve starts from top left of Cartesian plane. Label with the rule.	

UNDERSTANDING AND FLUENCY

5 Sketch the graph of each quartic relationship.

  a $y = (x - 1)(x + 2)(x + 3)(x - 3)$ **b** $y = (x + 4)(x + 1)(x - 1)(x + 3)$

  c $y = x(x - 2)(x - 6)(x + 5)$ **d** $y = -(x - 3)(x + 3)(x + 1)(x - 1)$

  e $y = (2x - 3)(x + 2)(x + 1)(x - 4)$ **f** $y = -(3x - 1)(x + 3)(x + 1)(x + 2)$

6 Sketch the graph of each polynomial relationship. (Hint: you will need to first factorise the polynomial.)

  a $y = x^3 - 3x^2 - 13x + 15$ **b** $y = -x^3 - 2x^2 + 16x + 32$

  c $y = x^3 - 2x^2 - 3x$ **d** $y = -2x^3 + 8x^2 - 2x - 12$

  e $y = x^4 - 15x^2 - 10x + 24$ **f** $y = x^4 - 9x^3 + 6x^2 + 56x$

  g $y = -x^4 + 10x^2 - 9$ **h** $y = -2x^4 - 9x^3 + 18x^2 + 71x + 30$

7 Use digital technology to check your graphs in question 6.

PROBLEM SOLVING AND REASONING

8 Consider the relationship $y = (x + 2)(x - 3)^2$.

a What type of polynomial relationship is this?

b Find the x- and y-intercepts.

c As the leading coefficient of the polynomial is positive, should the graph start from the top left or bottom left of the Cartesian plane?

d Sketch the graph. Use digital technology to verify your answer.

e What effect does the repeated factor of $(x - 3)$ have on the graph?

9 Sketch the graph of each cubic relationship.

a $y = (x + 4)(x - 1)^2$

b $y = (x + 2)^2(x - 5)$

c $y = -(x - 2)(x - 7)(x - 2)$

d $y = -(2x + 1)(x - 4)^2$

10 Sketch the graph of each quartic relationship.

a $y = (x - 3)(x + 2)(x + 1)^2$

b $y = -(x - 2)^2(x - 4)(x + 2)$

c $y = -x(x - 2)(x + 3)^2$

d $y = x^2(x + 5)(x - 3)$

e $y = (x - 1)^2(x + 2)^2$

f $y = -x^2(x - 4)^2$

11 The effect of having a repeated factor in a polynomial relationship means that an x-intercept is also a turning point. Investigate the effect of having a cubed repeated factor in a quartic relationship. That is, find what happens at $x = a$ for the graph of the form $y = (x - a)^3(x - b)$. Use digital technology and try different values for a and b.

12 A water ride can be modelled by the polynomial relationship $h = -\frac{1}{5}(t^3 - 11t^2 + 39t - 45)$, where h is the height above the ground in metres and t is the time in seconds from the start of the ride.

a Sketch a graph of this relationship.

b At what height above the ground does a person start the ride?

c How high is a person after 1 second?

d The ride descends to its lowest point. How long does this take?

e How long does it take for the ride to ascend and then descend again to its lowest point?

13 Amelia monitors the change in value of shares over the month of June. She finds that the dollar change in value, y, after x days can be approximated by a cubic relationship, where y increases from \$0 to \$21 after 5 days and is zero again after 12 days and 20 days.

a Sketch a graph of this cubic relationship.

b Find the rule for this relationship.

c Use this relationship to estimate the change in value at the end of June.

CHALLENGE

14 Sketch the graph of each polynomial relationship, clearly labelling all intercepts.

a $y = (x - 1)(x + 2)(x + 1)(x - 4)(x + 5)$

b $y = -x(x - 6)(x + 3)(x - 2)^2$

15 Use digital technology to produce the graphs in question 14 and locate and identify the coordinates of the turning points.

Reflect

What are the key features used to sketch a polynomial relationship?

6F Polynomials and transformations

Start thinking!

Graphs of polynomial relationships can also be transformed by performing dilation, reflection, translation or a combination of these.

1 Consider the graph of $y = x^3$. For each of the transformations listed below:

i describe how the graph of $y = x^3$ will be changed

ii sketch the new graph that is produced

iii write the rule for the new graph.

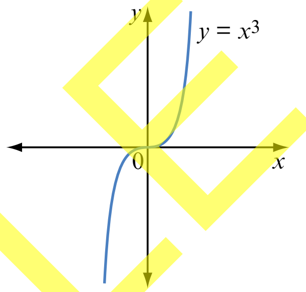
a dilate by a factor of 2

b reflect in the x-axis

c translate 3 units up

d translate 2 units right

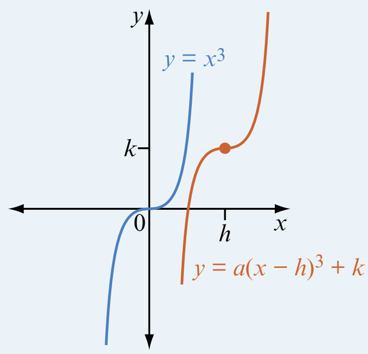
e translate 1 unit left and 4 units down



2 Repeat question **1** for the graph of $y = x^4$.

KEY IDEAS

- Transformations such as dilation, reflection and translation can be performed on the graph of $y = P(x)$.
  - Dilation by a factor of a produces $y = aP(x)$.
  - Reflection in the x-axis produces $y = -P(x)$.
  - Vertical translation of k units produces $y = P(x) + k$.
  - Horizontal translation of h units produces $y = P(x - h)$.



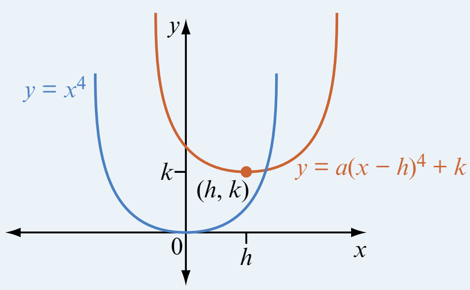
- A combination of transformations can be performed on the graph of $y = P(x)$ to produce the graph of $y = aP(x - h) + k$.

$$y = aP(x - h) + k$$

dilation by factor of a
For $a < 0$, reflection in x-axis.

horizontal translation of h units
For $h > 0$, move right.
For $h < 0$, move left.

vertical translation of k units
For $k > 0$, move up.
For $k < 0$, move down.

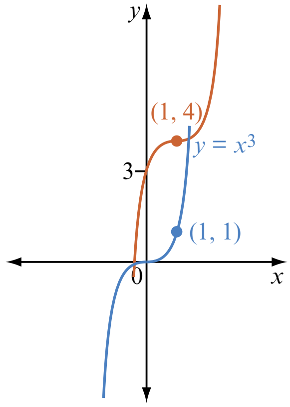


EXERCISE 6F Polynomials and transformations

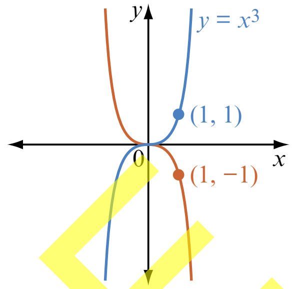
UNDERSTANDING AND FLUENCY

1 Identify the transformations performed on the graph of $y = x^3$ to produce each graph shown in orange.

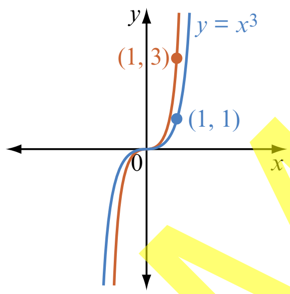
a



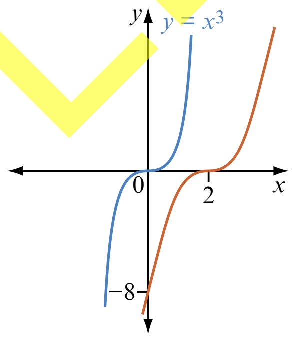
b



c



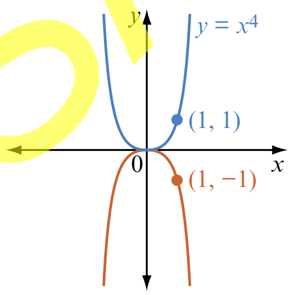
d



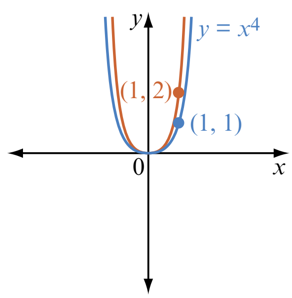
2 Write the rule for each graph in question **1**.

3 Identify the transformations performed on the graph of $y = x^4$ to produce each graph shown in orange.

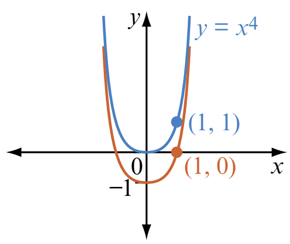
a



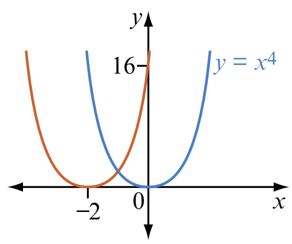
b



c



d



4 Write the rule for each graph in question **3**.

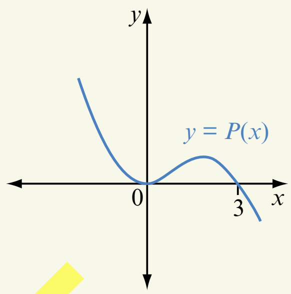
EXAMPLE 6F-1 Performing transformations on the graph of a polynomial relationship

Perform a transformation on the graph of $y = P(x)$ to produce the graph of:

a $y = P(x) + 2$

b $y = P(x + 2)$

c $y = -P(x)$.



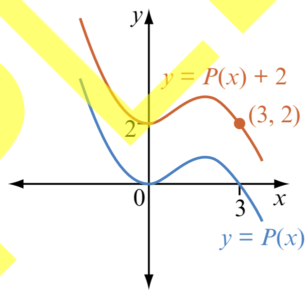
THINK

a **1** Identify the transformation.

2 No dilation or reflection so the shape remains the same. Move the original graph 2 units up. (0, 0) moves to (0, 2) and (3, 0) moves to (3, 2). Draw $y = P(x)$ and $y = P(x) + 2$ on the same Cartesian plane.

WRITE

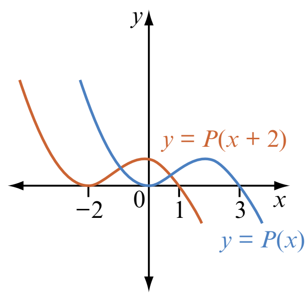
a Graph of $y = P(x)$ to be translated 2 units up.



b **1** Identify the transformation.

2 No dilation or reflection so the shape remains the same. Move the original graph 2 units left. (0, 0) moves to (−2, 0) and (3, 0) moves to (1, 0). Draw $y = P(x)$ and $y = P(x + 2)$ on the same Cartesian plane.

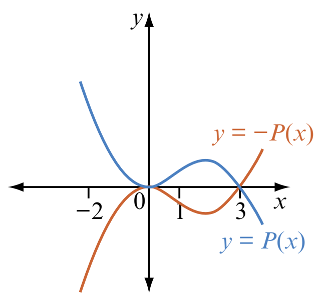
b Graph of $y = P(x)$ to be translated 2 units left.



c **1** Identify the transformation.

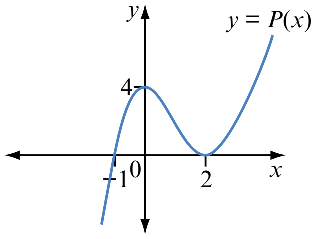
2 No dilation or translation. Draw $y = P(x)$ and $y = -P(x)$ on the same Cartesian plane.

c Graph of $y = P(x)$ to be reflected in the x-axis.



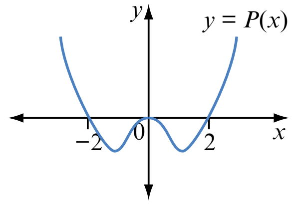
UNDERSTANDING AND FLUENCY

5 Perform a transformation on the graph of $y = P(x)$ to produce the graph of:



a $y = P(x) + 1$ b $y = P(x) - 2$
c $y = P(x - 1)$ d $y = P(x + 3)$
e $y = -P(x)$ f $y = 2P(x)$

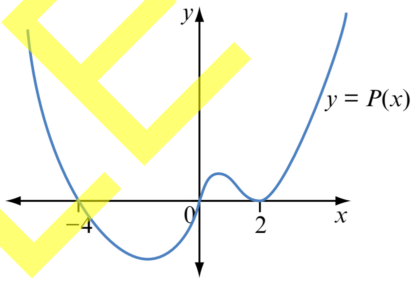
6 Perform a transformation on the graph of $y = P(x)$ to produce the graph of:



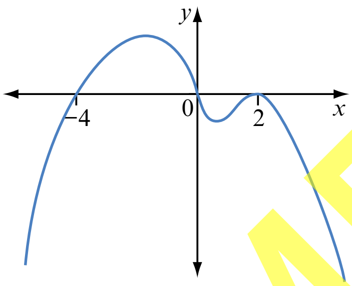
a $y = P(x) + 3$ b $y = P(x) - 1$
c $y = P(x - 3)$ d $y = P(x + 4)$
e $y = -P(x)$ f $y = 2P(x)$

PROBLEM SOLVING AND REASONING

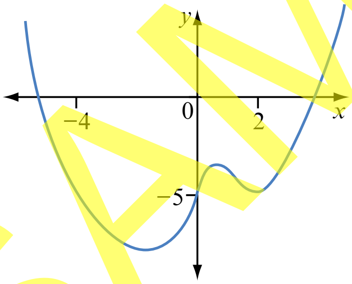
7 Identify the rule for each graph in terms of $P(x)$ using the graph of $y = P(x)$ shown at right.



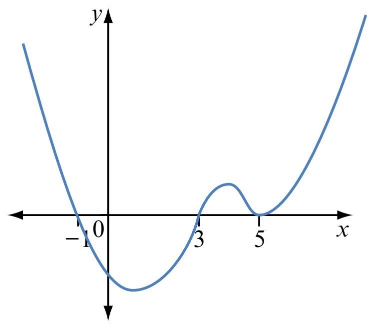
a



b



c



8 Compare the graph produced after reflecting $y = x^3$ in the x-axis with the graph produced after reflecting $y = x^3$ in the y-axis. What is the rule for each transformed graph?

CHALLENGE

9 For each cubic relationship:
 i describe the transformations to be performed on $y = x^3$ to produce the graph of the relationship
 ii identify the coordinates of the point of inflection
 iii find the x- and y-intercepts
 iv sketch the graph.

a $y = \frac{1}{2}(x - 3)^3 + 4$ b $y = -2(x + 1)^3 - 2$

10 Use digital technology to verify your answers to question 9.

Reflect

How are transformations useful when sketching a polynomial relationship?

CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

polynomial	dividend	factor theorem	turning point
leading term	divisor	polynomial relationship	transformations
leading coefficient	quotient	cubic relationship	dilation
constant	remainder	point of inflection	reflection
degree of polynomial	remainder theorem	quartic relationship	translation

MULTIPLE-CHOICE

6A 10A **1** Which expression is a polynomial?

A $x^2 + \sqrt{x}$ B $x^3 - 2x$
C $\frac{3x}{x^2 + 1}$ D $x^2 + x^{-1}$

6A 10A **2** The degree of the polynomial $4x^2 - 3x^4 + x^3 - 2x$ is:

A 1 B 2 C 3 D 4

6A 10A **3** The coefficient of the leading term in the expression $3 - 2x^2 + 5x^3 + 7x$ is:

A 3 B −2 C 5 D 7

6B 10A **4** When $x^2 + 6x - 3$ is divided by $x - 2$, the remainder is:

A 13 B −19 C 5 D −11

6B 10A **5** Which of these is correct?

A $(5x^2 - x + 2) \div (x - 4) = 5x + 19$ remainder -74
B $(3x^2 - 2x + 1) \div (x - 1) = 3x - 5$ remainder -9
C $(4x^2 - x + 8) \div (x - 2) = 4x - 9$ remainder -10
D $(2x^2 - x + 1) \div (x - 5) = 2x + 9$ remainder 46

6C 10A **6** When $P(x) = x^3 - 2x^2$ is divided by $(x - 1)$, the remainder is:

A $P(-1)$ B $P(1)$
C $P(-2)$ D $P(2)$

6C 10A **7** Which expression is a factor of $x^3 - 4x^2 + x + 6$?

A $x - 1$ B $x - 2$
C $x + 3$ D $x + 6$

6D 10A **8** The leading term in a polynomial is raised to the power n. The maximum number of solutions it could have is:

A n B $n + 1$
C $n - 1$ D impossible to tell

6D 10A **9** How many different solutions does $(x + 1)(x - 1)(x + 2)(x + 1) = 0$ have?

A 1 B 2 C 3 D 4

6E 10A **10** The graph of $y = (x - 3)(x + 2)(x - 5)$ has a y-intercept of:

A 3 B −2 C 5 D 30

6E 10A **11** The graph of $y = (x - 1)^3$ has a point of inflection at:

A $x = 1$ B $x = -1$
C $x = 0$ D $y = 1$

6F 10A **12** The transformation performed on the graph of $y = x^3$ to produce the graph of $y = (x + 5)^3$ is a translation of:

A 5 units right B 5 units left
C 5 units up D 5 units down

SHORT ANSWER

6A 10A **1** Decide if each expression is a polynomial. For each polynomial, give its name as linear, quadratic, cubic or quartic.

a $5 - 3x^2 + 4x - 6x^3$ b $1 - 3x$
c $\frac{4x}{5}$ d $\frac{6}{7x}$
e $6x^2 - 5x + 12$ f $\frac{1}{2}x - 3x^4$

6A 10A **2** For the polynomial $3x^4 + 2x^6 - 4x^5 + x - 8x^3 - 7x^2 - 5x^8$, identify:

a the number of terms
b the degree of the polynomial
c the constant term
d the leading term
e the leading coefficient
f the coefficient of the x^2 term

6B 10A **3** Use long division to find the quotient and remainder for each problem.

a $(x^2 + 5x - 2) \div (x - 2)$
b $(3x^2 - x + 4) \div (x + 1)$
c $(x^2 - x + 8) \div (x - 4)$
d $(x^3 + 2x^2 - x + 3) \div (x^2 + 1)$

6B 10A **4** Expand and simplify the right side to verify whether each statement is true.

a $2x^2 + 5x - 1 = (2x + 3)(x + 1) - 4$
b $x^2 - 3x + 5 = (x - 1)(x - 2) + 7$

6C 10A **5** Use the factor theorem to find a linear factor of each polynomial $P(x)$.

a $P(x) = x^3 - x^2 - 5x - 3$
b $P(x) = x^3 + 4x^2 + 7x + 12$
c $P(x) = x^3 + 3x^2 - 6x + 2$
d $P(x) = x^3 + x^2 + 4$

6C 10A **6** Fully factorise each polynomial.

a $x^3 - 7x^2 + 14x - 8$
b $x^3 - 7x - 6$

6C 10A **7** Two factors of the polynomial $x^3 - 39x - 70$ are $(x - 7)$ and $(x + 5)$. What is the third factor?

6D 10A **8** Solve each equation.

a $(x - 5)(x + 2)(x - 1)(x + 1) = 0$
b $x(x + 3)(x - 3)(x - 2) = 0$
c $(x + 3)^2(x - 5)^2 = 0$
d $x^2(x + 4)(x - 4) = 0$

6D 10A **9** Fully factorise the left side of each equation, then solve.

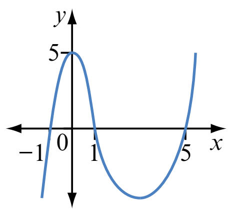
a $x(x - 2)(x^2 - 4) = 0$
b $(x - 4)(x^2 - 8x + 16) = 0$

6E 10A **10** Sketch the graphs of:

a $y = (x - 3)(x + 1)(x - 2)$
b $y = -(x + 2)^2(x - 4)$.

6E 10A **11** For the cubic relationship shown,

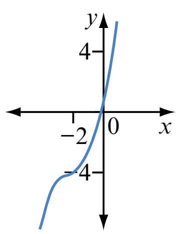
a identify the x-intercepts
b identify the y-intercept
c write the rule as a product of factors.



6E 10A **12** Consider $y = x^4 - 5x^2 + 4$.

a Write the relationship in terms of its factors.
b Identify the x- and y-intercepts
c Sketch the graph.

6F 10A **13** Describe the transformation/s performed on the graph of $y = x^3$ to produce the graph shown.

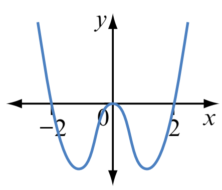


6F 10A **14** Consider the graph of $y = x(x - 1)^3$.

a Identify the x- and y-intercepts.
b Sketch the graph.

6F 10A **15** Write the rule for this polynomial relationship in:

a factor form
b expanded form.



MIXED PRACTICE

10A **1** If $P(x) = 4x^3 - x^2 + 2x - 1$, the value of $P(-1)$ is:

A -8 **B** 2 **C** 0 **D** 4

10A **2** If $P(x) = 2x^3 - 3x^2 + 5x - 4$, calculate values for:

a $P(2)$ **b** $P(-1)$ **c** $P(-2)$
d $P(-3)$ **e** $P(4)$ **f** $P(0)$

10A **3** The solution to $x^3 - 19x + 30 = 0$ is:

A $x = -5, -2$ or 3 **B** $x = -5, 2$ or 3
C $x = 5, -2$ or 3 **D** $x = 5, 2$ or -3

10A **4** If $P(x) = 3x^3 - 4x^2 + x$ and $Q(x) = -x^3 + 2x^2 - 4$, evaluate:

a $P(-1)$ **b** $Q(-1)$
c $P(x) + Q(x)$ **d** $Q(x) - P(x)$
e $P(0) - Q(0)$ **f** $P(1) - 4Q(1)$

10A **5** Use long division to find the quotient and remainder for each problem.

a $(x^3 - 4x^2 + 5x - 8) \div (x - 2)$
b $(x^3 - 2x^2 - 3x + 4) \div (x - 1)$
c $(x^3 + 3x^2 + 8x - 5) \div (x - 3)$

10A **6** Expand and simplify each product.

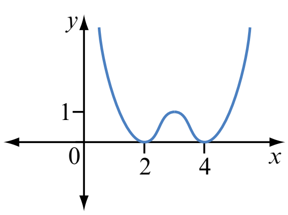
a $3x^2(4x - 8)$ **b** $(x^3 - 5x)^2$
c $(x^4 - 3x^2)(5x^2 + 2x^4)$ **d** $(x^2 - 1)^2(x + 1)$

10A **7** The graph of the polynomial $y = (x + 3)^2(x - 3)^2$ has how many solutions?

A 1 **B** 2
C 3 **D** 4

10A **8** Write the rule for this graph in:

a factor form
b expanded form.



10A **9** Decide if $(x + 2)$ is a factor of:

a $x^3 - 3x^2 - 6x + 8$
b $x^3 - 4x^2 - 3x + 18$
c $x^3 + 4x^2 - 3x - 18$

10A **10** The expansion of $(1 - x)^3$ is:

A $3x^2 - x^3 + 1 - 3x$
B $1 - 2x + 2x^2 + x^3$
C $1 - x^3$
D $3 - 3x - x^3$

10A **11** The graph of $y = -(x + 5)(x - 3)(x - k)$ has a y-intercept of 30. What is the value of k?

10A **12** The division of the polynomial $4x^3 - 2x^2 - x + 10$ by $(x - 2)$ gives a quotient and remainder of:

A $4x^2 - 4x + 7$ remainder -4
B $4x^2 + 6x - 13$ remainder -16
C $4x^2 + 6x + 11$ remainder 32
D $4x^2 + 4x - 7$ remainder 4

10A **13** One linear factor of $x^3 - 7x + 6$ is:

A $x + 3$ **B** $x + 2$
C $x + 1$ **D** $x + 4$

10A **14** The polynomial $x^3 - 2x^2 - 5x + k$ is divisible by $(x - 1)$, $(x + 2)$ and $(x - 3)$. What is the value of the constant k?

A -1 **B** 2 **C** -3 **D** 6

10A **15** One factor of the polynomial $x^3 - 43x - 42$ is $(x + 1)$. What are the other factors?

10A **16** Consider $y = (x + 1)^2(x - 3)^2$.

a What type of polynomial relationship is this?
b Find the x- and y-intercepts.
c As the leading coefficient of the polynomial is positive, should the graph start from the top left or bottom left of the Cartesian plane?
d What effect does the repeated factors of $(x + 1)$ and $(x - 3)$ have on the graph?

10A **17** Give all the x-intercepts of the graph of the polynomial $P(x) = x(x + 1)(x - 4)(x^2 - 4)$.

10A **18** Fully factorise the polynomial $x^4 - 2x^2 + 1$.

10A **19** Find the solution to the equation $-3x^3 + 9x^2 + 30x - 72 = 0$.

ANALYSIS

The infinity symbol

You are familiar with the infinity symbol – it looks like the number 8 on its side. Its shape is like that of a cubic relationship with three x-intercepts.

a It is possible to model the infinity sign using two cubic relationships graphed for the same set of x values.

i Complete this table for $y = x^3 - x$.

x	−1	−0.6	−0.5	0	0.5	0.6	1
y							

ii Repeat part **i** for $y = -x^3 + x$.

iii On a Cartesian plane, plot the points in the tables from parts **i** and **ii**. Join the points with a smooth curve.

iv Describe the set of x values used for these two graphs.

v Do the turning points for the two relationships occur at $x = -0.5$ and 0.5? Explain.

vi Use digital technology to draw the graphs of these two relationships for the set of x values described in part **iv**. Find the coordinates of the turning points.

b It is also possible to model the infinity symbol by tracing the path a point takes as it moves from an angle of 0° with the x-axis to an angle of 360° with the x-axis.

The coordinates of all points (x, y) on the infinity symbol can be represented by the values $\left(\cos\theta, \frac{\sin 2\theta}{2}\right)$, where θ is the angle the line joining the point with the origin makes with the x-axis.

i Construct a table for x and y using angles of θ every 15° from 0° to 360°. (This number of points is necessary to get an accurate representation of the graph.)

Angle (θ)	x ($\cos\theta$)	y $\left(\frac{\sin 2\theta}{2}\right)$
0°		
15°		
30°		
…		
345°		
360°		

ii Complete the table, giving your answers correct to two decimal places, if necessary. You may find some of the trigonometric values negative as the angle increases beyond 90°. You will understand the reason for this when you study the trigonometric ratios for angles greater than 90° in Chapter 8.

iii Plot the points (x, y) on a Cartesian plane. Join them with a smooth curve. (Alternatively, you could use a spreadsheet to plot the graph.)

iv Describe the shape of the graph.

v Describe the set of x and y values used for the graph.

c Compare the two models. Do you consider one to be a better model of the infinity symbol than the other? Explain.

CONNECT

Modelling a roller coaster ride

Engineers use polynomials to model roller coaster rides. Relationships can be formed for the height of the ride after a given time. For example, one relationship where h is the height in metres of a roller coaster after t seconds is

$h = -0.5t^6 + 5t^5 - 13.75t^4 - 5t^3 + 63.5t^2 - 59t + 20$.

You can use digital technology to produce its graph.

In this task, you will be looking at a few simpler relationships to model sections of roller coaster rides.

Your task

You are to complete the following three problems. Include all necessary graphs and working to justify your answers.

Problem 1

There are three rollercoaster rides at a fun park. Each can be represented by a polynomial relationship with height h in metres after t seconds.

- Ride of terror: $h = -0.1t^3 + 1.8t^2 - 9.6t + 16$
- Fear factor: $h = 0.3t^3 - 5t^2 + 21t$
- Ride of your life: $h = -2t^4 + 21t^3 - 61t^2 + 36t + 36$

Describe each ride to your friend who will be visiting the fun park the next day. Include information such as the initial height of the ride, times when the ride skims the ground or goes through an underground tunnel and the realistic duration of the ride. Use a sketch graph to help you.

Your friend is nervous of extreme heights. Use digital technology to find the maximum height of each ride.

Problem 2

A new roller coaster ride is to be designed so that, after completing a loop, it moves up from ground level, skims the ground again after a further 3 seconds and finishes on the ground after 7 seconds. It needs to have two thrilling ‘up and down’ sections. Determine a suitable polynomial relationship to model the conditions of this ride after completing the loop. Explain your reasoning.

Problem 3

Design your own roller coaster ride. Use a polynomial of degree 4 or higher and fully explain how you modelled the relationship. Describe this ride to your friend.

As an extension, choose another scenario that could be modelled by a polynomial relationship of degree 3 or higher. Instead of using time as the independent variable, you may like to use a length or distance variable. Include all reasoning, working and diagrams.

You may like to present your findings as a report. Your report could include:

- a poster showing diagrams and calculations
- a PowerPoint presentation
- a technology demonstration
- other (check with your teacher).